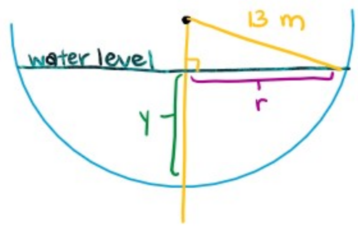
Related Rates Example

Thursday, June 15, 2023 2:24 AM

Water is flowing at a rate of $6\ m^3/min$ from a reservoir shaped like a hemispherical bowl of radius 13 m. Given $V = \frac{\pi}{3} y^2 (3R - y)$, with R the radius of the bowl and y the depth of the water in meters, answer the following questions.

What we know:

- $\frac{dV}{dt} = -6\ m^3/min$
- $R = 13\ m$
- $V = \frac{\pi}{3} y^2 (3R - y)$



(a) At what rate is the water level changing when the water is $\underbrace{\text{8 m deep}}_{y=8}$?

We want to find $\frac{dy}{dt}$ when $y = 8$.

We know $\frac{dV}{dt}$, R, and V. Let's relate them to $\frac{dy}{dt}$. (Note that the formula for V involves a couple of y's. There is hope!)

$$V = \frac{\pi}{3} y^2 (3R - y) = \frac{\pi}{3} y^2 (3 \cdot 13 - y) = \frac{\pi}{3} y^2 (39 - y) = 13\pi y^2 - \frac{\pi}{3} y^3.$$

$$\frac{dV}{dt} = \frac{d}{dt}\left[13\pi y^2 - \frac{\pi}{3} y^3\right]$$

$$= 13\pi \frac{d}{dt}[y^2] - \frac{\pi}{3}\frac{d}{dt}[y^3]$$ constant multiple rule

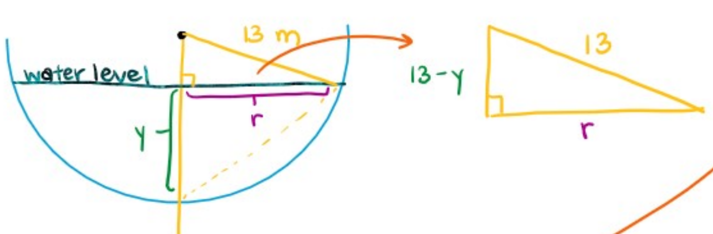
$$= 13\pi \cdot 2y \frac{dy}{dt} - \frac{\pi}{3} \cdot 3y^2 \cdot \frac{dy}{dt}$$ power rule + chain rule

$$-6 = \frac{dy}{dt}(26\pi y - \pi y^2)$$

$$\frac{dy}{dt} = -\frac{6}{26\pi y - \pi y^2}$$

$$\left.\frac{dy}{dt}\right|_{y=8} = -\frac{6}{26\pi(8) - \pi(8)^2} = -\frac{6}{208\pi - 64\pi} = -\frac{6}{144\pi} = -\frac{1}{24\pi}\ m/min$$

(b) What is the radius r of the water's surface when the water is y meters deep?



Here we have a right triangle whose dimensions relate y and r, as desired for this question.

Using the Pythagorean Theorem:

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$$13^2 = (13-y)^2 + r^2$$

$$\cancel{13^2} = \cancel{13^2} - 13y - 13y + y^2 + r^2$$

$$0 = -26y + y^2 + r^2$$

$$r^2 = 26y - y^2$$

$$\boxed{r = \sqrt{26y - y^2}}$$

(c) At what rate is radius r changing when the water is 8 m deep?

$\underbrace{\text{water is 8 m deep}}_{y=8}$

From part (b), $r = \sqrt{26y - y^2} = (26y - y^2)^{\frac{1}{2}}$, so:

$$\frac{dr}{dt} = \frac{1}{2}(26y - y^2)^{-\frac{1}{2}} \cdot (26 - 2y) \cdot \frac{dy}{dt}$$

known from part (a)

$$\frac{dr}{dt} = \frac{1}{2}(26y - y^2)^{-\frac{1}{2}} \cdot (26 - 2y) \cdot \left(-\frac{1}{24\pi}\right)$$

$$\left.\frac{dr}{dt}\right|_{y=8} = \frac{1}{2}(26\cdot 8 - 8^2)^{-\frac{1}{2}}(26 - 2\cdot 8)\cdot\left(-\frac{1}{24\pi}\right)$$

$$= \frac{1}{2}(208 - 64)^{-\frac{1}{2}} \cdot (26 - 16) \cdot \left(-\frac{1}{24\pi}\right)$$

$$= \frac{1}{2}(144)^{-\frac{1}{2}} \cdot (10) \cdot \left(-\frac{1}{24\pi}\right)$$

$$\frac{1}{144^{\frac{1}{2}}} = \frac{1}{\sqrt{144}}$$

$$= \frac{1}{2} \cdot \frac{1}{12} \cdot 10 \cdot \left(-\frac{1}{24\pi}\right)$$

$$= -\frac{10}{24 \cdot 24\pi}$$

$$= \boxed{-\frac{5}{288\pi} \text{ m/min}}$$