MA 2071 Matrices & Linear Algebra I Discussion 1

Worcester Polytechnic Institute

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Augmented Matrix

Suppose we have the following system of linear equations:

$$\begin{cases} x + 2z = 5\\ y - 30z = -16\\ x - 2y + 4z = 8 \end{cases}$$

The equivalent augmented matrix is:

$$\begin{bmatrix} 1 & 0 & 2 & & 5 \\ 0 & 1 & -30 & & -16 \\ 1 & -2 & 4 & & 8 \end{bmatrix}$$

Note: The submatrix to the left of the vertical line is the **coefficient matrix**.

Row Reduction

Properties of reduced echelon form (REF)

- 1. All nonzero rows are above any rows of all zeroes.
- 2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
- 3. All entries in a column below a leading entry are zero.

Properties of row reduced echelon form (RREF)

- 4. The leading entry in each row is 1.
- 5. Each leading 1 is the only nonzero entry in its column.

$$\begin{bmatrix} 1 & 0 & 2 & | & 5 \\ 0 & 1 & -30 & | & -16 \\ 0 & 0 & -58 & | & -29 \end{bmatrix} \iff \text{This is in reduced echelon form.}$$
$$\begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & \frac{1}{2} \end{bmatrix} \iff \text{This is in row reduced echelon form.}$$

The green numbers are the leading entries.

Definition: A **pivot position** is a location corresponding to a leading entry in the reduced form of the matrix.

Definition: A **pivot** is a nonzero number in a pivot position.

Pivots

Let's say you have a 3×4 coefficient matrix (three equations, four variables):



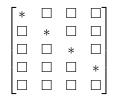
You can have at most three pivots:



Now, let's say you have a 5×4 coefficient matrix (five equations, four variables):



You can have at most four pivots:



Main Idea: If you have m equations and n variables, you will have a $m \times n$ coefficient matrix. The maximum number of pivots is $\min\{m, n\}$ (the smaller of m and n).

Other insights from pivots:

 In a consistent system with a unique solution, the last row of the row reduced echelon form of the augmented matrix will look like

$$\begin{bmatrix} 0 & \dots & 0 & 1 & | & b \end{bmatrix},$$

where $b \in \mathbb{R}$.

• In an inconsistent system, the last row of the row reduced echelon form of the augmented matrix will look like

$$\begin{bmatrix} 0 & \dots & 0 & 0 & | & b \end{bmatrix},$$

where $b \neq 0$.

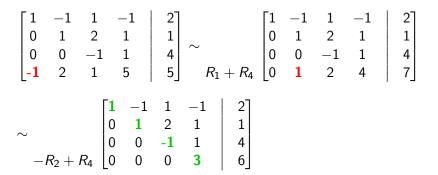
Example: Row Reduction

Suppose we have the following system of linear equations:

$$\begin{cases} x - y + z - w = 2\\ y + 2z + w = 1\\ -z + w = 4\\ -x + 2y + z + 5w = 5 \end{cases}$$

The equivalent augmented matrix is:

$$\begin{bmatrix} 1 & -1 & 1 & -1 & | & 2 \\ 0 & 1 & 2 & 1 & | & 1 \\ 0 & 0 & -1 & 1 & | & 4 \\ -1 & 2 & 1 & 5 & | & 5 \end{bmatrix}$$



We are now in reduced echelon form.

Notice: We have four variables and four pivots, so we know there is a unique solution!

For row reduced echelon form, we need each leading entry to be 1:

$$\begin{bmatrix} \mathbf{1} & -\mathbf{1} & \mathbf{1} & -\mathbf{1} & & | & 2 \\ 0 & \mathbf{1} & 2 & \mathbf{1} & & | & \mathbf{1} \\ 0 & 0 & -\mathbf{1} & \mathbf{1} & & | & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{3} & & | & \mathbf{6} \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & -\mathbf{1} & \mathbf{1} & -\mathbf{1} & & | & 2 \\ 0 & \mathbf{1} & 2 & \mathbf{1} & & | & \mathbf{1} \\ 0 & 0 & \mathbf{1} & -\mathbf{1} & & | & -\mathbf{4} \\ 0 & 0 & 0 & \mathbf{1} & & | & 2 \end{bmatrix}$$

Now, use multiples of the leading entries to eliminate the other nonzero numbers in the coefficient matrix, starting from the right.

$$\begin{bmatrix} 1 & -1 & 1 & -1 & | & 2 \\ 0 & 1 & 2 & 1 & | & 1 \\ 0 & 0 & 1 & -1 & | & -4 \\ 0 & 0 & 0 & 1 & | & 2 \end{bmatrix} \xrightarrow{R_4 + R_1} \begin{bmatrix} 1 & -1 & 1 & 0 & | & 4 \\ 0 & 1 & 2 & 0 & | & -1 \\ 0 & 0 & 1 & 0 & | & -2 \\ 0 & 0 & 0 & 1 & 0 & | & -2 \\ 0 & 0 & 0 & 1 & | & 2 \end{bmatrix}$$

$$\sim \xrightarrow{R_3 + R_1} \begin{bmatrix} 1 & -1 & 0 & 0 & | & 6 \\ 0 & 1 & 0 & 0 & | & 3 \\ 0 & 0 & 1 & 0 & | & -2 \\ 0 & 0 & 0 & 1 & | & 2 \end{bmatrix}$$

$$\sim \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 & | & 9 \\ 0 & 1 & 0 & 0 & | & 3 \\ 0 & 0 & 1 & 0 & | & -2 \\ 0 & 0 & 0 & 1 & | & 2 \end{bmatrix}$$

Thus, the solution is (x, y, z, w) = (9, 3, -2, 2).