MA 2071 Matrices & Linear Algebra I

Discussion 1

Worcester Polytechnic Institute

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[1] Slides created by Abigail Lindner

Augmented Matrix

Suppose we have the following system of linear equations:

$$\begin{cases} x + 2z = 5 \\ y - 30z = -16 \\ x - 2y + 4z = 8 \end{cases}$$

The equivalent **augmented matrix** is:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -30 & -16 \\ 1 & -2 & 4 & 8 \end{array}\right]$$

Note: The submatrix to the left of the vertical line is the **coefficient matrix**.

Row Reduction

Properties of reduced echelon form (REF)

1. All nonzero rows are above any rows of all zeroes.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

Properties of row reduced echelon form (RREF)

4. The leading entry in each row is 1.
5. Each leading 1 is the only nonzero entry in its column.

$$\left[\begin{array}{ccc|c} \mathbf{1} & 0 & 2 & 5 \\ 0 & \mathbf{1} & -30 & -16 \\ 0 & 0 & \mathbf{-58} & -29 \end{array}\right] \Longleftarrow \text{ This is in } \textbf{reduced echelon form}.$$

$$\left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & 4 \\ 0 & \mathbf{1} & 0 & -1 \\ 0 & 0 & \mathbf{1} & \frac{1}{2} \end{array}\right] \Longleftarrow \text{ This is in } \textbf{row reduced echelon form}.$$

The **green** numbers are the **leading entries**.

Definition: A **pivot position** is a location corresponding to a leading entry in the reduced form of the matrix.

Definition: A **pivot** is a nonzero number in a pivot position.

Pivots

Let's say you have a 3×4 coefficient matrix (three equations, four variables):

$$\begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix}$$

You can have *at most* three pivots:

$$\begin{bmatrix} * & \square & \square & \square \\ \square & * & \square & \square \\ \square & \square & * & \square \end{bmatrix}$$

Now, let's say you have a 5×4 coefficient matrix (five equations, four variables):

$$\begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix}$$

You can have *at most* four pivots:

$$\begin{bmatrix} * & \square & \square & \square \\ \square & * & \square & \square \\ \square & \square & * & \square \\ \square & \square & \square & * \\ \square & \square & \square & \square \end{bmatrix}$$

Main Idea: If you have m equations and n variables, you will have a $m \times n$ coefficient matrix. The maximum number of pivots is $\min\{m, n\}$ (the smaller of m and n).

Other insights from pivots:

- In a consistent system with a unique solution, the last row of the row reduced echelon form of the augmented matrix will look like
$$\begin{bmatrix} 0 & \dots & 0 & 1 & | & b \end{bmatrix},$$
where $b \in \mathbb{R}$.
- In an inconsistent system, the last row of the row reduced echelon form of the augmented matrix will look like
$$\begin{bmatrix} 0 & \dots & 0 & 0 & | & b \end{bmatrix},$$
where $b \neq 0$.

Example: Row Reduction

Suppose we have the following system of linear equations:

$$\begin{cases} x - y + z - w = 2 \\ y + 2z + w = 1 \\ -z + w = 4 \\ -x + 2y + z + 5w = 5 \end{cases}$$

The equivalent **augmented matrix** is:

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & -1 & 1 & 4 \\ -1 & 2 & 1 & 5 & 5 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & -1 & 1 & 4 \\ \mathbf{-1} & 2 & 1 & 5 & 5 \end{array}\right] \sim_{R_1 + R_4} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & -1 & 1 & 4 \\ 0 & \mathbf{1} & 2 & 4 & 7 \end{array}\right]$$

$$\sim_{-R_2 + R_4} \left[\begin{array}{cccc|c} \mathbf{1} & -1 & 1 & -1 & 2 \\ 0 & \mathbf{1} & 2 & 1 & 1 \\ 0 & 0 & \mathbf{-1} & 1 & 4 \\ 0 & 0 & 0 & \mathbf{3} & 6 \end{array}\right]$$

We are now in **reduced echelon form**.

Notice: We have four variables and four pivots, so we know there is a unique solution!

For row reduced echelon form, we need each **leading entry** to be 1:

$$\left[\begin{array}{cccc|c} \mathbf{1} & -1 & 1 & -1 & 2 \\ 0 & \mathbf{1} & 2 & 1 & 1 \\ 0 & 0 & \mathbf{-1} & 1 & 4 \\ 0 & 0 & 0 & \mathbf{3} & 6 \end{array}\right] \sim \begin{array}{c} \\ \\ -R_3 \\ \frac{1}{3}R_4 \end{array} \left[\begin{array}{cccc|c} \mathbf{1} & -1 & 1 & -1 & 2 \\ 0 & \mathbf{1} & 2 & 1 & 1 \\ 0 & 0 & \mathbf{1} & -1 & -4 \\ 0 & 0 & 0 & \mathbf{1} & 2 \end{array}\right]$$

Now, use multiples of the leading entries to eliminate the other nonzero numbers in the coefficient matrix, starting from the right.

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \sim \begin{array}{c} R_4 + R_1 \\ -R_4 + R_2 \\ R_4 + R_3 \\ \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 4 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\sim \begin{array}{c} -R_3 + R_1 \\ -2R_3 + R_2 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\sim \begin{array}{c} R_2 + R_1 \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Thus, the solution is $(x, y, z, w) = (9, 3, -2, 2)$.